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Slow flow of a fluid carrying a uniform current past a conducting ellipsoid of revolution

By C. SOZOU
Department of Applied Mathematics and Computing Science,
The University, Sheffield

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The method of matched asymptotic expansions is employed for investigating the effect of a uniform current on the velocity field of a viscous, incompressible, conducting fluid streaming past a stationary conducting ellipsoid of revolution, assuming that the Reynolds number is small. It is also assumed that at infinity the velocity and uniform current are parallel to the axis of the ellipsoid. It is found that the presence of the current increases or decreases the drag coefficient, depending on whether the fluid conductivity is larger than that of the ellipsoid or vice versa. It is suggested that this effect of the current on the drag coefficient holds for all axisymmetric bodies that are also symmetric about a plane perpendicular to their axis. The case of a circular disk broadside on the undisturbed current, obtained as a special case of a planetary ellipsoid, is slightly different; when the conductivity of the disk is non-zero the electromagnetically induced flow field vanishes.

1. Introduction

The Stokes creeping flow, induced by the distortion of a uniform current, in a conducting fluid occupying the whole space, by the presence of a non-conducting sphere, was investigated by Chow (1966). The present author (Sozou 1970) extended this work to the case when the distorting non-conducting body is an ellipsoid of revolution with its axis parallel to the direction of the undisturbed current at infinity. In both these papers it was found that the induced flow velocity was finite and non-zero at infinity and the drag was not affected by the induced velocity field. It was suggested that the non-vanishing of the induced velocity at infinity was due to the use of Stokes approximation, that is, the neglect of the inertia terms from the momentum equation.

Chow & Billings (1967), retaining the inertia terms in the momentum equation, investigated the problem of a current carrying fluid streaming past a stationary sphere, by using the method of matched asymptotic expansions. This method, a summary of which may be found in Van Dyke's (1964) book, was developed by Kaplun & Lagerstrom (1957) and Proudman & Pearson (1957). Here, we also retain the inertia terms in the momentum equation and, using the same method of matched asymptotic expansions, we investigate the problem of a current carrying fluid streaming past a stationary conducting ellipsoid of revolution, having its axis parallel to the direction of the undisturbed stream at infinity. The problem considered by Chow & Billings (1967) and that of a circular disk broadside on to the direction of the undisturbed current, are special cases of the more general problem considered here. The corresponding problem for a non-conducting fluid was considered by Breach (1961).

2. Flow past an ovary ellipsoid

We use a cylindrical co-ordinate system (r, θ, x) with the origin at the centre of the stationary ellipsoid and the x axis along the axis of symmetry of the ellipsoid, which is parallel to the direction of the undisturbed current and stream at infinity. At infinity the current density is J_0 and the fluid speed is U_0.

If we let

$$\mathbf{V} = \frac{1}{r}\left(-\frac{\partial}{\partial x}, 0, \frac{\partial}{\partial r}\right)\psi$$

be the fluid velocity and take the curl of the momentum equation we have

$$\frac{\hat{\boldsymbol{\theta}}}{r^2} D_0\psi\, D^2\psi = \frac{\hat{\boldsymbol{\theta}}}{r}\nu D^4\psi - \frac{1}{\rho}\nabla\times(\mathbf{J}\times\mathbf{B}), \tag{1}$$

where $\mathbf{J}$ is the current density, $\mathbf{B}$ the magnetic field, ρ the fluid density, ν the coefficient of kinematic viscosity,

$$D^2 = \frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial x^2}-\frac{1}{r}\frac{\partial}{\partial r} \quad \text{and} \quad D_0\psi = \frac{2}{r}\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial r}.$$

At infinity $\mathbf{J}$ tends to its undisturbed value $\mathbf{J}_0$. At the surface of the body, J_n, the normal component of $\mathbf{J}$, and the tangential component of the electric field are continuous. The current $\mathbf{J}$ and the associated magnetic field are given in the author's earlier work (Sozou 1970) for the case when $J_n = 0$. When the ellipsoid is conducting, $J_n \neq 0$ and the corresponding expressions for $\mathbf{J}$ and $\mathbf{B}$ must be modified by replacing the parameter B_0 in the original work by b_0, where

$$b_0 = 2(\sigma_0-\sigma_1)\bigg/\left\{\sigma_0\left[\log\left(\frac{1+e}{1-e}\right)-2e\right]+\sigma_1\left[\frac{2e}{1-e^2}-\log\left(\frac{1+e}{1-e}\right)\right]\right\}$$

and σ_0 and σ_1 are the electrical conductivities of the ellipsoid and the fluid, respectively.

Let a be the semi-major axis and e the eccentricity of a meridian section of the ellipsoid. If we use ellipsoidal co-ordinates ζ and $\mu\,(=\cos\theta)$ we have

$$x = ae\mu\zeta, \quad r = ae(1-\mu^2)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}} \tag{2}$$

and the ellipsoid is given by $\zeta = \zeta_0 \geqslant 1$. When $\zeta_0 = 1$ we have an elongated rod.

We take a characteristic length a and define non-dimensional (primed) variables by

$$x = ax', \quad r = ar', \quad \psi = U_0a^2\psi'.$$

If we replace B_0 by b_0 in the $\mathbf{J}\times\mathbf{B}$ expression computed by the author (Sozou

1970), and use (2), the non-dimensional variables and omit primes, (1) becomes,

$$E_1\psi E^2\psi = -\frac{e}{R}E^4\psi + b_0 e^6 R_1 \frac{\mu(1-\mu^2)}{\zeta^2-\mu^2}\left[1-\tfrac{1}{2}b_0\left\{\log\left(\frac{\zeta+1}{\zeta-1}\right)-\frac{2\zeta}{\zeta^2-1}\right\}\right], \tag{3}$$

where R is the Reynolds number given by $R = aU_0/\nu$,

$$R_1 = \frac{4\pi J_0^2 a^2}{\rho U_0^2} = \frac{16\pi^2 J_0^2 a^2}{8\pi}\Big/\tfrac{1}{2}\rho U_0^2 \sim \frac{B^2}{8\pi}\Big/\tfrac{1}{2}\rho U_0^2,$$

$$E^2 = \frac{1}{\zeta^2-\mu^2}\left[(\zeta^2-1)\frac{\partial^2}{\partial\zeta^2}+(1-\mu^2)\frac{\partial^2}{\partial\mu^2}\right],$$

and

$$E_1\psi = -\frac{1}{\zeta^2-\mu^2}\left[\frac{\partial\psi}{\partial\mu}\left(\frac{2\zeta}{\zeta^2-1}-\frac{\partial}{\partial\zeta}\right)+\frac{\partial\psi}{\partial\zeta}\left(\frac{2\mu}{1-\mu^2}+\frac{\partial}{\partial\mu}\right)\right].$$

(i) *Oseen expansion far from the ellipsoid*

We introduce the Oseen variable

$$z = R\zeta, \tag{4}$$

where z is fixed as R tends to zero and set

$$\Psi = \frac{\Psi_0(z,\mu)}{R^2}+\frac{\Psi_1(z,\mu)}{R}+\dots, \tag{5}$$

where the leading term in (5) represents a uniform stream at infinity and is given by

$$\Psi_0 = \tfrac{1}{2}e^2z^2(1-\mu^2). \tag{6}$$

Substituting (4), (5) and (6) in (3) we obtain a linear equation in Ψ_1 the solution of which is

$$\begin{aligned}\Psi_1 &= -C_1(1+\mu)\left[1-\exp\{-\tfrac{1}{2}ez(1-\mu)\}\right]-\tfrac{1}{6}b_0 e^4 R_1 z(1-\mu^2)\\ &= -C_1(1+\mu)\left[1-\exp\{-\tfrac{1}{2}eR\zeta(1-\mu)\}\right]-\tfrac{1}{6}b_0 e^4 R R_1 \zeta(1-\mu^2),\end{aligned} \tag{7}$$

where C_1 is an arbitrary constant which will be determined later.

(ii) *Stokes expansion near the ellipsoid*

For the inner region we set

$$\psi = \psi_0(\zeta,\mu)+R\psi_1(\zeta,\mu)+\dots, \tag{8}$$

where ψ_0 represents a uniform stream past the ellipsoid and is given by (Happel & Brenner 1965)

$$\psi_0 = \tfrac{1}{2}e^2(\zeta^2-1)(1-\mu^2)\left[1-\frac{\dfrac{\zeta_0^2+1}{\zeta_0^2-1}\log\left(\dfrac{\zeta+1}{\zeta-1}\right)-\dfrac{2\zeta}{\zeta^2-1}}{\dfrac{\zeta_0^2+1}{\zeta_0^2-1}\log\dfrac{\zeta_0+1}{\zeta_0-1}-\dfrac{2\zeta_0}{\zeta_0^2-1}}\right]. \tag{9}$$

If we substitute (8) and (9) in (3) we obtain

$$E^4\psi_1+e^{-1}E_1\psi_0E^2\psi_0 = \frac{b_0e^5R_1}{\zeta^2-\mu^2}\left[1-\tfrac{1}{2}b_0\left\{\log\left(\frac{\zeta+1}{\zeta-1}\right)-\frac{2\zeta}{\zeta^2-1}\right\}\right]\mu(1-\mu^2). \tag{10}$$

The solution of this equation is

$$\psi_1 = C_2\psi_0 + \frac{b_0 e^5 R_1}{7A_0}[f_0(\zeta) - f_0(\zeta_0) - (7\mu^2-3)f_1(\zeta)]\mu(1-\mu^2) + (1-\mu^2)\sum_1^\infty X_{2m}(\zeta)P'_{2m}(\mu), \qquad (11)$$

where C_2 is an arbitrary constant and the expression in square brackets is the solution obtained by the author (Sozou 1970) when the inertia terms were neglected. A_0, $f_0(\zeta)$ and $f_1(\zeta)$ are specified in the author's original paper. The expression

$$(1-\mu^2)X_{2m}(\zeta)P'_{2m}(\mu)$$

is a particular integral of the equation

$$E^4\psi_1 + e^{-1}E_1\psi_0 E^2\psi_0 = 0$$

and was worked out by Breach (1961). $P'_n(\mu)$ denotes the first derivative of the Legendre polynomial of degree n. $X_{2m}(\zeta)$ is a complicated function which for large ζ is $O(\zeta^{4-2m})$. Here we use a slightly different notation and absorb a constant in the function $X_{2m}(\zeta)$ as originally defined by Breach (1961).

For large ζ

$$f_0(\zeta) \sim \frac{7A_0\zeta^2}{24}, \quad f_1(\zeta) \sim \frac{A_0}{120}, \quad X_2(\zeta) \sim -\frac{C_0 e^5\zeta^2}{72}, \qquad (12)$$

where

$$C_0 = 12\Big/\left[(1+e^2)\log\left(\frac{1+e}{1-e}\right) - 2e\right].$$

If we expand the Oseen and, making use of (12), the Stokes expression for ψ in ascending powers of the Oseen variable z we have

$$\psi - \text{Oseen} = \frac{e^2z^2(1-\mu^2)}{2R^2} - \frac{C_1 ez}{2R}(1-\mu^2) - \frac{1}{6R}b_0 e^4 R_1 z(1-\mu^2) + \frac{C_1}{8R}e^2z^2(1-\mu^2)(1-\mu)$$

and

$$\psi - \text{Stokes} = \frac{e^2z^2(1-\mu^2)}{2R^2} - \frac{C_0 e^4}{6R}z(1-\mu^2) + \frac{C_2 e^2}{2R}z^2(1-\mu^2) + \frac{b_0 e^5 R_1}{24R}z^2\mu(1-\mu^2) - \frac{C_0 e^5}{24R}z^2\mu(1-\mu^2).$$

Hence, for matching

$$C_1 = \tfrac{1}{3}e^3(C_0 - b_0 R_1) \quad \text{and} \quad C_2 = \tfrac{1}{12}e^3(C_0 - b_0 R_1).$$

3. Flow past a planetary ellipsoid

Let c be the semi-major axis and e the eccentricity of a meridian section of the ellipsoid. If we use ellipsoidal co-ordinates μ and ζ we have

$$x = ce\mu\zeta, \quad r = ce(\zeta^2+1)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}$$

and the ellipsoid is given by $\zeta = \zeta_0 \geqslant 0$. The case $\zeta_0 = 0$ corresponds to a circular disk.

If we do an analysis similar to that of the last section and express the stream function in units of U_0c^2 we find

$$\psi-\text{Oseen} = \tfrac{1}{2}e^2\zeta^2(1-\mu^2)-c_1(1+\mu)\left[1-\exp\{-\tfrac{1}{2}R\zeta e(1-\mu)\}\right]-\tfrac{1}{6}b_1R_1e^4\zeta(1-\mu^2), \tag{13}$$

$$\psi-\text{Stokes} = (1+\tfrac{1}{4}Rc_1)\psi_0+\frac{RR_1b_1e^5}{7A_1}\left[g_0(\zeta)-g_0(\zeta_0)+(7\mu^2-3)g_1(\zeta)\right]\mu(1-\mu^2)$$
$$+(1-\mu^2)\sum_1^\infty Z_{2m}(\zeta)P'_{2m}(\mu), \tag{14}$$

where R is the Reynolds number defined by $R = cU_0/\nu$,

$$R_1 = 4\pi J_0^2c^2/\rho U_0^2, \quad c_1 = e^3(c_0-b_1R_1)/3,$$

$$c_0 = 6/[e(1-e^2)^{\frac{1}{2}}-(1-2e^2)\sin^{-1}e],$$

$$b_1 = (\sigma_0-\sigma_1)/\{\sigma_0[e(1-e^2)^{-\frac{1}{2}}-\sin^{-1}e]+\sigma_1[\sin^{-1}e-e(1-e^2)^{\frac{1}{2}}]\},$$

σ_0 and σ_1 are the respective conductivities of the ellipsoid and fluid, and

$$\psi_0 = \tfrac{1}{2}e^3(\zeta^2+1)(1-\mu^2)\left[1-\frac{\dfrac{\zeta}{\zeta^2+1}-\dfrac{\zeta_0^2-1}{\zeta_0^2+1}\cot^{-1}\zeta}{\dfrac{\zeta_0}{\zeta_0^2+1}-\dfrac{\zeta_0^2-1}{\zeta_0^2+1}\cot^{-1}\zeta_0}\right].$$

The expression in square brackets in (14) is the solution obtained by the author (Sozou 1970) when the inertia terms were ignored, with B_1 being replaced by b_1, which is the corresponding parameter for the case when the ellipsoid is conducting. A_1, $g_0(\zeta)$ and $g_1(\zeta)$ are defined in the author's original paper. The expression

$$(1-\mu^2)\Sigma Z_{2m}(\zeta)P'_{2m}(\mu)$$

is analogous to the expression

$$(1-\mu^2)\Sigma X_{2m}(\zeta)P'_{2m}(\mu)$$

of the last section and the functions $Z_{2m}(\zeta)$ can be worked out from the functions $X_{2m}(\zeta)$ (Breach 1961).

The case of a sphere can be obtained from that of an ovary or that of a planetary ellipsoid by letting e tend to zero and ζ to infinity in such a way that $e\zeta$ tends to r, the distance from the centre of the sphere.

4. Drag coefficients

The contribution to the drag coefficient comes from ψ_0 in the Stokes expansion (note that ψ_1 has a ψ_0 component) of the stream function. All the other terms of the expansion are odd in μ and thus they do not contribute to the drag.

For an ovary ellipsoid the drag coefficient is found to be

$$\frac{4\pi C_0e^3}{3R}(1+\tfrac{1}{4}RC_1). \tag{15}$$

For a planetary ellipsoid the drag coefficient is

$$\frac{4\pi c_0 e^3}{3R}(1+\tfrac{1}{4}Rc_1). \tag{16}$$

If we let e tend to zero, either in (15) or (16), we obtain

$$\frac{6\pi}{R}[1+\tfrac{1}{8}R(3+\sigma R_1)],$$

which is the drag coefficient for a sphere. Here

$$\sigma = 2(\sigma_1-\sigma_0)/(2\sigma_1+\sigma_0),\ R = aU_0/\nu \quad \text{and} \quad R_1 = 4\pi J_0^2 a^2/\rho U_0^2,$$

where a is the radius of the sphere.

From (15), (16) and the expressions for C_1, c_1, b_0 and b_1, it follows that the presence of the current increases the drag coefficient if σ_1 is greater than σ_0. When σ_0 is greater than σ_1 the current decreases the drag coefficient. This can be explained as follows:

When $\sigma_1 > \sigma_0$ the ellipsoid offers a greater resistance than the fluid to the flow of current. The current lines are bent away from the axis of the ellipsoid and the current flux through the ellipsoid is less than when $\sigma_0 = \sigma_1$. The rotational part of the Lorentz force, which produces the induced flow field, is proportional to its component parallel to the axis of the ellipsoid. This component is directed towards the body and is symmetrical with respect to the equatorial plane of the ellipsoid but increases the coefficient of the 'oseenlet', of the inner solution, which is proportional to the drag coefficient. When $\sigma_0 > \sigma_1$ the picture is reversed. The current lines are bent towards the axis of the ellipsoid and the current flux through the ellipsoid is greater than when $\sigma_1 = \sigma_0$. The rotational part of the Lorentz force is directed away from the ellipsoid and decreases the drag coefficient.

More generally, when a fluid carrying a uniform current is streaming past an axisymmetric body with steady low velocity which at infinity is parallel to the axis of the body, the bending of the current lines will be away or towards the body axis and the rotational part of the Lorentz force will be parallel to the axis and directed towards or away from the body, depending on whether the fluid conductivity σ_1 is greater or less than the body conductivity σ_0. If the body has also symmetry about a plane perpendicular to its axis, the Lorentz force is symmetrical about this plane and, like the case of the ellipsoid, the drag coefficient increases or decreases depending on whether σ_1 is greater than σ_0 or vice versa. Indeed, one is tempted to conjecture that the drag coefficient increases when $\sigma_1 > \sigma_0$ and decreases when $\sigma_0 > \sigma_1$ irrespective of whether the body has a plane of symmetry perpendicular to its axis.

The drag coefficient for a circular disk perpendicular to the undisturbed stream is derived from that of a planetary ellipsoid by letting e tend to 1. When $\sigma_0 = 0$, c_0 and c_1 become $12/\pi$ and $2(R_1+6)/3\pi$, respectively, and the drag coefficient for the non-conducting disk comes to be

$$\frac{8}{3\pi R}[6\pi + R(R_1+6)].$$

When $\sigma_0 \neq 0$, as e tends to 1, b_1, that is, the effect of the current on the drag coefficient tends to zero. This is to be expected. If the disk is non-conducting the current lines bend round to avoid it. When, however, the disk is conducting the current lines go straight through it and the current is undisturbed. This is true for any conducting lamina, plane or curved, and is due to the boundary conditions, which imply that, when the conducting obstacle is a lamina, **J** is continuous. Thus, the effect of the current on the drag coefficient of a thin body is minimal if the body is conducting and could be appreciable if the body is non-conducting.

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